Bipartite graphs with uniquely restricted maximum matchings and their corresponding greedoids

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Abstract

A *maximum stable set* in a graph G is a stable set of maximum size. S is a *local maximum stable set* of G, and we write $S \in \Psi(G)$, if S is a maximum stable set of the subgraph spanned by $S \cup N(S)$, where $N(S)$ is the neighborhood of S. A matching M is *uniquely restricted* if its saturated vertices induce a subgraph which has a unique perfect matching, namely M itself. Nemhauser and Trotter Jr. [12], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of G. In [10] we have shown that the family $\Psi(T)$ of a forest T forms a greedoid on its vertex set. In this paper we demonstrate that for a bipartite graph $G, \Psi(G)$ is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. If $X, Y \subset V$ are disjoint and non-empty, then by (X, Y) we mean the set $\{xy : xy \in E, x \in XA, y \in Y\}$. The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \quad and \ vw \in E\}$. If $|N(v)| = 1$, then v is a *pendant vertex* of G; by $\text{pend}(G)$ we designate the set of all pendant vertices of G. We denote the *neighborhood* of $A \subset V$ by $N_G(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and its *closed neighborhood* by $N_G[A] = A \cup N(A)$, or shortly, $N(A)$ and $N[A]$, if no ambiguity. K_n, C_n denote respectively, the complete graph on $n \geq 1$ vertices and the chordless cycle on $n \geq 3$ vertices. By $G = (A, B, E)$ we mean a bipartite graph having $\{A, B\}$ as its standard bipartition.

A *stable* set in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a *maximum stable set* of G, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G. Let $\Omega(G)$ stand for the set of all maximum stable sets of G. A set $A \subseteq V(G)$ is a *local maximum stable set* of G if A is a maximum stable set in the subgraph spanned by $N[A]$, i.e., $A \in \Omega(G[N[A]])$, [10]. In the sequel, by $\Psi(G)$ we denote the set of all local maximum stable sets of the graph G. For instance, any set $S \subseteq \mathrm{pend}(G)$ belongs to $\Psi(G)$, while the converse is not generally true; e.g., $\{a\}, \{e, d\} \in \Psi(G)$ and $\{e, d\} \cap \mathrm{pend}(G) = \emptyset$, where G is the graph in Figure 1.

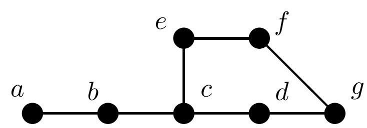

Figure 1: A graph with diverse local maximum stable sets.

Not any stable set of a graph G is included in some maximum stable set of G. For example, there is no $S \in \Omega(G)$ such that $\{c, f\} \subset S$, where G is the graph depicted in Figure 4. The following theorem due to Nemhauser and Trotter Jr. [12], shows that some special maximum stable sets can be enlarged to maximum stable sets.

Theorem 1.1 *[12] Any local maximum stable set of a graph is a subset of a maximum stable set.*

Let us notice that the converse of Theorem 1.1 is not generally true. For instance, C_n, $n \geq 4$, has no proper local maximum stable set. The graph G in Figure 1 shows another counterexample: any $S \in \Omega(G)$ contains some local maximum stable set, but these local maximum stable sets are of different cardinalities. As examples, $\{a, d, f\} \in \Omega(G)$ and $\{a\}, \{d, f\} \in \Psi(G)$, while for $\{b, e, g\} \in \Omega(G)$ only $\{e, g\} \in \Psi(G)$.

In [10] we have proved the following result:

Theorem 1.2 *The family of local maximum stable sets of a forest of order at least two forms a greedoid on its vertex set.*

Theorem 1.2 is not specific for forests. For instance, the family $\Psi(G)$ of the graph G in Figure 2 is a greedoid.

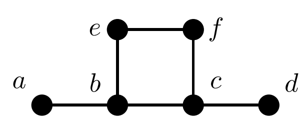

Figure 2: A graph whose family of local maximum stable sets forms a greedoid.

The definition of greedoids we use in the sequel is as follows.

Definition 1.3 *[1], [6] A greedoid is a pair $(E, \mathcal{F})$, where $\mathcal{F} \subseteq 2^E$ is a set system satisfying the following conditions:*
(Accessibility) for every non-empty $X \in \mathcal{F}$ there is an $x \in X$ such that $X - \{x\} \in \mathcal{F}$;
(Exchange) for $X, Y \in \mathcal{F}, |X| = |Y|+1$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

Clearly, $\Omega(G) \subseteq \Psi(G)$ holds for any graph G. It is worth observing that if $\Psi(G)$ is a greedoid and $S \in \Psi(G)$, $|S| = k \geq 2$, then by accessibility property, there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, ..., x_{k-1}\} \subset \{x_1, ..., x_{k-1}, x_k\} = S$$

such that $\{x_1, x_2, ..., x_j\} \in \Psi(G)$, for all $j \in \{1, ..., k-1\}$. Such a chain we call an *accessibility chain* for S. As an example, for $S = \{a, c, e\} \in \Psi(G)$, where G is the graph in Figure 2, an accessibility chain is $\{a\} \subset \{a, e\} \subset S$.

A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ having the property that no two edges of M share a common vertex. We denote the size of a *maximum matching* (a matching of maximum cardinality) by $\mu(G)$. A *perfect matching* is a matching saturating all the vertices of the graph.

Let us recall that G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$, [2], [7]. As a well-known example, any bipartite graph is a König-Egerváry graph. Some non-bipartite König-Egerváry graphs are presented in Figure 7.

A matching $M = \{a_i b_i : a_i, b_i \in V(G), 1 \leq i \leq k\}$ of a graph G is called *a uniquely restricted matching* if M is the unique perfect matching of the subgraph $G[\{a_i, b_i : 1 \leq i \leq k\}]$, [4] (first time this kind of matching appeared in [5] for bipartite graphs under the name "*constrained matching*". Let $\mu_r(G)$ be the maximum size of a uniquely restricted matching in G. Clearly, $0 \leq \mu_r(G) \leq \mu(G)$ holds for any graph G. For instance, $0 = \mu_r(C_{2n}) < n = \mu(C_{2n})$, while $\mu_r(C_{2n+1}) = \mu(C_{2n+1}) = n$.

In this paper we characterize the bipartite graphs whose family of local maximum stable sets are greedoids. Namely, we prove that for a bipartite graph G, the family $\Psi(G)$ is a greedoid on the vertex set of G if and only if all its maximum matchings are uniquely restricted.

Golumbic, Hirst and Lewenstein have shown in [4] that $\mu_r(G) = \mu(G)$ holds when G is a tree or has only odd cycles. Our findings reveal another class of graphs enjoying this property.

2 Preliminary results

An edge e of a graph G is *α-critical* (*μ-critical*) if $\alpha(G) < \alpha(G-e)$ ($\mu(G) > \mu(G-e)$, respectively). Let us observe that there is no general connection between the α-critical and the μ-critical edges of a graph. For instance, the edge e of the graph G_1 in Figure 3 is μ-critical and non-α-critical, while the edge e of the graph G_2 in the same figure is α-critical and non-μ-critical.

Nevertheless, for König-Egerváry graphs and especially for bipartite graphs, there is a closed relationship between these two kinds of edges.

Lemma 2.1 *[11] In a König-Egerváry graph, α-critical edges are also μ-critical, and they coincide in a bipartite graph.*

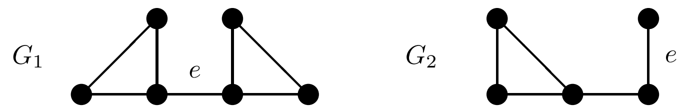

Figure 3: Non-König-Egervary graphs.

In a König-Egerváry graph, maximum matchings have a very specific property, emphasized by the following statement:

Lemma 2.2 *[9] Any maximum matching M of a König-Egerváry graph G is contained in each $(S, V(G) - S)$ and $|M| = |V(G) - S|$, where $S \in \Omega(G)$.*

Clearly, not any matching of a graph is contained in a maximum matching. For example, there is no maximum matching of the graph G in Figure 2 that includes the matching $M = \{ab, cf\}$. Let us observe that M is a maximum matching in $G[N[\{a, f\}]], \{a, f\}$ is stable in G, but $\{a, f\} \notin \Psi(G)$. The following result shows that, under certain conditions, a matching of a bipartite graph can be extended to a maximum matching.

Lemma 2.3 *If G is a bipartite graph, $\widehat{S} \in \Psi(G)$, and $\widehat{M}$ is a maximum matching in $G[N[\widehat{S}]]$, then there exists a maximum matching M in G such that $\widehat{M} \subseteq M$.*

Proof. Let $W = N(\widehat{S})$, $H = G[N[\widehat{S}]]$, and S' be a stable set in G such that $S = \widehat{S} \cup S' \in \Omega(G)$ (such S' exists according to Theorem 1.1). Since H is bipartite and $\widehat{M}$ is a maximum matching in H, it follows that

$$\left|\widehat{S}\right| + \left|\widehat{M}\right| = \alpha(H) + \mu(H) = |V(H)| = \left|\widehat{S}\right| + |W|.$$

Let M be a maximum matching in G. Then, by Lemma 2.2, $M \subseteq (S, V(G) - S)$, because $S \in \Omega(G)$, and

$$|M| = |V(G) - S| = \left|N(\bar{S})\right| + \left|N(S') - N(\widehat{S})\right| = \left|\widehat{M}\right| + |V(G) - S - W|.$$

Let M' be the subset of M containing edges having an endpoint in $V(G) - S - W$. Since no edge joins a vertex of $\widehat{S}$ to some vertex in $V(G) - S - W$, it follows that M' is the restriction of M to $G[V(G) - S - W]$. Consequently, $\widehat{M} \cup M'$ is a matching in G that contains $\widehat{M}$, and because $\left|\widehat{M} \cup M'\right| = \left|\widehat{M}\right| + |V(G) - S - W| = |M|$, we see that $\widehat{M} \cup M'$ is a maximum matching in G. ■

Let us notice that Lemma 2.3 can not be generalized to non-bipartite graphs. For instance, the graph G presented in Figure 4 has $\widehat{S} = \{a, d\} \in \Psi(G), \widehat{M} = \{ac, df\}$ is a maximum matching in $G[N[\widehat{S}]]$, but there is no maximum matching in G that includes $\widehat{M}$.

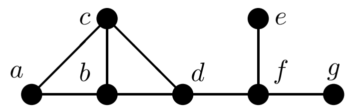

Figure 4: $\widehat{M} = \{ac, df\}$ is a maximum matching in $G[N[\{a, d\}]]$.

Lemma 2.4 *If $G = (A, B, E)$ is a connected bipartite graph having a unique perfect matching, then $A \cap \operatorname{pend}(G) \neq \emptyset$ and $B \cap \operatorname{pend}(G) \neq \emptyset$.*

Proof. Let $M = \{a_i b_i : 1 \leq i \leq n, a_i \in A, b_i \in B\}$ be the unique perfect matching of G. Clearly, $|A| = |B|$. Suppose that $B \cap \operatorname{pend}(G) = \emptyset$. Hence, $|N(b_i)| \geq 2$ for any $b_i \in B$.

Under these conditions, we shall build some cycle C having half of edges contained in M, and this allows us to find a new perfect matching in G, which contradicts the uniqueness of M. We begin with the edge $a_1 b_1$. Since $|N(b_1)| \geq 2$, there is some $a \in (A - \{a_1\}) \cap N(b_1)$, say a_2. We continue with $a_2 b_2 \in M$. Further, $N(b_2)$ contains some $a \in (A - \{a_2\})$. If $a_1 \in N(b_2)$, we are done, because $G[\{a_1, a_2, b_1, b_2\}] = C_4$. Otherwise, we may suppose that $a = a_3$, and we add to the growing cycle the edge $a_3 b_3$. Since G has a finite number of vertices, after a number of edges from M, we must find some edge $a_j b_k$ with $1 \leq j < k$. So, the cycle C we found has

$$V(C) = \{a_i, b_i : j \leq i \leq k\},\ E(C) = \{a_i b_i : j \leq i \leq k\} \cup \{b_i a_{i+1} : j \leq i < k\} \cup \{a_j b_k\}.$$

Clearly, half of edges of C are contained in M.

Similarly, we can show that also $A \cap \operatorname{pend}(G) \neq \emptyset$. ■

The following proposition presents a recursive structure of bipartite graphs owing unique perfect matchings, which generalizes the recursive structure of trees having perfect matching due to Fricke, Hedetniemi, Jacobs and Trevisan, [3].

Proposition 2.5 *K_2 is a bipartite graph, and it has a unique perfect matching. If G is a bipartite graph with a unique perfect matching, then $G + K_2$ is also a bipartite graph having a unique perfect matching. Moreover, any bipartite graph containing a unique perfect matching can be obtained in this way.*

By $G + K_2$ we mean the graph comprising the disjoint union of G and K_2, and additional edges joining at most one of endpoints of K_2 to vertices belonging to only one color class of G.

Proof. Let $G = (A, B, E)$ be a bipartite graph having a unique perfect matching, say $M = \{a_i b_i : 1 \leq i \leq n, a_i \in A, b_i \in B\}$. If $K_2 = (\{x, y\}, \{xy\})$, then $H = G + K_2$ is also bipartite and $M \cup \{xy\}$ is a unique perfect matching in H, since M was unique in G and at least one of x, y is pendant in H.

Conversely, let G be a bipartite graph with a unique perfect matching. By Lemma 2.4, it follows that G has at least one pendant vertex, say x. If $y \in N(x)$, then, clearly, $G = (G - \{x, y\}) + K_2$. ■

3 Main results

Proposition 3.1 *If G is a bipartite graph of order $2n$ having a perfect matching M, then M is unique if and only if for some $S \in \Omega(G)$ there exists an accessibility chain.*

Proof. Since $\mu(G) = n$, in every set of size greater than n there exists a pair of adjacent vertices, and hence $\alpha(G) = n$.

Suppose that G is a bipartite graph of order $2n$ with a unique perfect matching. We prove, by induction on n, that for some $S \in \Omega(G)$ there exists an accessibility chain.

For $n = 2$, let $S = \{x_1, x_2\} \in \Omega(G), N(S) = \{y_1, y_2\}$ and $x_1y_1, x_2y_2 \in M$, where M is its unique perfect matching. Then, at least one of x_1, x_2 is pendant, say x_1. Hence, $\{x_1\} \subset \{x_1, x_2\} = S$ is an accessibility chain.

Suppose that the assertion is true for $k < n$. Let $G = (A, B, E)$ be of order $2n$ and $M = \{a_ib_i : 1 \leq i \leq n, a_i \in A, b_i \in B\}$ be its unique perfect matching. According to Proposition 2.5, $G = H + K_2$. Consequently, we may assume that: $K_2 = (\{a_1, b_1\}, \{a_1b_1\})$ and $a_1 \in \text{pend}(G)$. Clearly, H is a bipartite graph containing a unique perfect matching, namely $M_H = M - \{a_1b_1\}$.

Case 1. $a_1 \in S$. Hence, $S_{n-1} = S - \{a_1\} \in \Omega(H)$, and by induction hypothesis, there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{n-2}\} \subset \{x_1, x_2, ..., x_{n-1}\} = S_{n-1}$$

such that $\{x_1, x_2, ..., x_k\} \in \Psi(H)$ for any $k \in \{1, ..., n-1\}$. Since $N(a_1) = \{b_1\}$, it follows that $N_G(\{x_1, x_2, ..., x_k\} \cup \{a_1\}) = N_H(\{x_1, x_2, ..., x_k\}) \cup \{b_1\}$, and therefore $\{x_1, x_2, ..., x_k\} \cup \{a_1\} \in \Psi(G)$ for any $k \in \{1, ..., n-1\}$. Clearly, $\{a_1\} \in \Psi(G)$, and consequently, we have the chain:

$$\begin{aligned} \{a_1\} &\subset \{a_1, x_1\} \subset \{a_1, x_1, x_2\} \subset ... \subset \{a_1, x_1, x_2, ..., x_{n-2}\} \subset \\ &\subset \{a_1, x_1, x_2, ..., x_{n-1}\} = \{a_1\} \cup S_{n-1} = S, \end{aligned}$$

where $\{a_1, x_1, x_2, ..., x_k\} \in \Psi(G)$, for all $k \in \{1, ..., n-1\}$.

Case 2. $b_1 \in S$. Hence, $S_{n-1} = S - \{b_1\} \in \Omega(H)$ and also $S_{n-1} \in \Psi(G)$, because $N_G[S_{n-1}] = A \cup B - \{a_1, b_1\}$. By induction hypothesis, there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{n-2}\} \subset \{x_1, x_2, ..., x_{n-1}\} = S_{n-1}$$

such that $\{x_1, x_2, ..., x_k\} \in \Psi(H)$ for any $k \in \{1, ..., n-1\}$. Since none of a_1, b_1 is contained in $N_G(\{x_1, x_2, ..., x_k\})$, it follows that $\{x_1, x_2, ..., x_k\} \in \Psi(G)$, for any $k \in \{1, ..., n-1\}$. Consequently, we have the chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{n-1}\} = S_{n-1} \subset S_{n-1} \cup \{b_1\} = S,$$

where $\{x_1, x_2, ..., x_k\} \in \Psi(G)$, for all $k \in \{1, ..., n-1\}$.

Conversely, let $M = \{x_iy_i : 1 \leq i \leq n\}$ be a perfect matching in G, and suppose that for $S \in \Omega(G)$ there exists a chain of local maximum stable sets

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{n-1}\} \subset \{x_1, x_2, ..., x_{\alpha-1}, x_\alpha\} = S.$$

We show, by induction on $k = |\{x_1, x_2, ..., x_k\}|$ that $H_k = G[N[\{x_1, x_2, ..., x_k\}]]$ owns a unique perfect matching.

For $k = 1$, the assertion is true, because $\{x_1\} \in \Psi(G)$ ensures that x_1 is pendant, and therefore, $H_1 = G[N[\{x_1\}]]$ has a unique perfect matching, consisting of the unique edge issuing from x_1, namely $x_1 y_1$.

Assume that H_k has a unique perfect matching, say M_k. We may assert that $M_k \subseteq M$, because M_k is unique and included in H_k and also M matches $x_1, x_2, ..., x_k$ onto vertices belonging to $N(\{x_1, x_2, ..., x_k\})$. Hence, $M_{k+1} = M_k \cup \{x_{k+1}y_{k+1}\}$ is a maximum matching in H_{k+1}. If M_{k+1} is not unique in H_{k+1}, then there exists some $z \in N(a_{k+1}) - N[\{a_1, a_2, ..., a_k\}]$ such that $z \neq y_{k+1}$. Therefore, we infer that the set $\{x_1, x_2, ..., x_k\} \cup \{z, y_{k+1}\}$ is stable in H_{k+1} and larger than $\{x_1, x_2, ..., x_{k+1}\}$, which contradicts the fact that $\{x_1, x_2, ..., x_{k+1}\} \in \Psi(G)$. Consequently, M_{k+1} is unique and also perfect in H_{k+1}. ■

If one of the maximum matchings of a bipartite graph is uniquely restricted, this is not necessarily true for all its maximum matchings. For instance, let us consider the bipartite graph G presented in Figure 5. The set of edges $M_1 = \{ab, ce\}$ is one of uniquely restricted maximum matchings of G, while $M_2 = \{bd, cf\}$ is one of its maximum matchings, but it is not uniquely restricted.

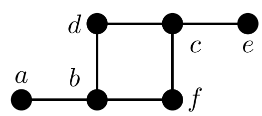

Figure 5: Not all maximum matchings of a graph have to be uniquely restricted.

Theorem 3.2 *If G is a bipartite graph, then the following assertions are equivalent:*
(i) there exists some $S \in \Omega(G)$ having an accessibility chain;
(ii) there exists a uniquely restricted maximum matching in G;
(iii) each $S \in \Omega(G)$ has an accessibility chain.

Proof. $(i) \Rightarrow (ii)$ Let us consider an accessibility chain of $S \in \Omega(G)$

$$\emptyset \subset \{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{\alpha-1}\} \subset \{x_1, x_2, ..., x_\alpha\} = S,$$

for which we define $S_i = \{x_1, x_2, ..., x_i\}$ and $S_0 = \emptyset$.

Since $S_{i-1} \in \Psi(G), S_i = S_{i-1} \cup \{x_i\} \in \Psi(G)$ and G is bipartite, it follows that $|N(x_i) - N[S_{i-1}]| \leq 1$, because otherwise, if $\{a, b\} \subset N(x_i) - N[S_{i-1}]$, then the set $\{a, b\} \cup S_{i-1}$ is stable in $N[S_{i-1} \cup \{x_i\}]$, and larger than $S_i = S_{i-1} \cup \{x_i\}$, in contradiction with the fact that $S_i \in \Psi(G)$.

Let $\{y_{i_j} : 1 \leq j \leq \mu\}$ be such that $\{y_{i_j}\} = N(x_{i_j}) - N[S_{i_j - 1}]$, for all $i \in \{1, ..., \alpha\}$ with $|N(x_i) - N[S_{i-1}]| = 1$. Hence, $M = \{x_{i_j} y_{i_j} : 1 \leq j \leq \mu\}$ is a matching in G.

- *Claim 1.* $\mu = \mu(G)$, i.e., M is a maximum matching in G.

Since $|N(x_i) - N[S_{i-1}]| \leq 1$ holds for all $i \in \{1, ..., \alpha\}$, where $S_0 = N[S_0] = \emptyset$, and $\{y_{i_j}\} = N(x_{i_j}) - N[S_{i_j - 1}]$, for all $i \in \{1, ..., \alpha\}$ satisfying $|N(x_i) - N[S_{i-1}]| = 1$, it follows that $N(S) = \{y_{i_j} : 1 \leq j \leq \mu\}$, and this ensures that M is a maximal matching in G, i.e., it is impossible to add an edge to M and to get a new matching.

In addition, we have

$$|V(G)| = |N[S]| = |S| + |N(S)| = |S| + \left|\{y_{i_j} : 1 \leq j \leq \mu\}\right| = \alpha(G) + |M|,$$

and because $|V(G)| = \alpha(G) + \mu(G)$, we infer that $|M| = \mu(G)$. In other words, M is a maximum matching in G.

- *Claim 2.* M is a uniquely restricted maximum matching in G.

We use induction on $k = |S_k|$ to show that the restriction of M to $H_k = G(N[S_k])$, which we denote by M_k, is a uniquely restricted maximum matching in H_k.

For $k = 1, S_1 = \{x_1\} \in \Psi(G)$ and this implies that $N(x_1) = \{y_{i_1}\}$. Clearly, $M_1 = \{x_1 y_{i_1}\}$ is a uniquely restricted maximum matching in H_1.

Suppose that the assertion is true for all $j \leq k - 1$. Let us observe that

$$N[S_k] = N[S_{k-1}] \cup (N(x_k) - N[S_{k-1}]) \cup \{x_k\},$$

because $S_k = S_{k-1} \cup \{x_k\}$.

Further we will distinguish between two different situations depending on the number of new vertices, which the set $N(x_k)$ brings to the set $N[S_{k-1}]$.

Case 1. $N(x_k) - N[S_{k-1}] = \emptyset$. Hence, we obtain:

$$|V(H_k)| = |S_{k-1} \cup \{x_k\}| + |M_{k-1}| = |S_k| + |M_{k-1}| = \alpha(H_k) + |M_{k-1}|.$$

Since $|V(H_k)| = \alpha(H_k) + \mu(H_k)$, the equality $|V(H_k)| = \alpha(H_k) + |M_{k-1}|$ ensures that M_{k-1} is a maximum matching of H_k. Therefore, M_{k-1} is a uniquely restricted maximum matching in H_k.

Case 2. $N(x_k) - N[S_{k-1}] = \{y_{i_k}\}$. Then we have:

$$|V(H_k)| = |S_{k-1} \cup \{x_k\}| + |M_{k-1} \cup \{x_k y_{i_k}\}| = |S_k| + |M_k| = \alpha(H_k) + |M_k|,$$

and this assures that $M_k = M_{k-1} \cup \{x_k y_{i_k}\}$ is a maximum matching in H_k. The edge $e = x_k y_{i_k}$ is α-critical in H_k, because $\{y_{i_k}\} = N(x_k) - N[S_{k-1}]$. According to Lemma 2.1, e is also μ-critical in H_k. Therefore, any maximum matching of H_k contains e, and since $M_k = M_{k-1} \cup \{e\}$ and M_{k-1} is a uniquely restricted maximum matching in $H_{k-1} = H_k - \{x_k, y_{i_k}\}$, it follows that M_k is a uniquely restricted maximum matching in H_k.

(ii) $\Rightarrow$ *(iii)* Let M be a uniquely restricted maximum matching in G. According to Lemma 2.2, $M \subseteq (S, V(G) - S)$ and $|M| = |V(G) - S| = \mu(G)$. Therefore, M is a unique perfect matching in $H = G[N[S_\mu]]$, where

$$S_\mu = \{x : x \in S, x \textit{ is an endpoint of an edge in } M\}.$$

It is clear that S_μ is a maximum stable set in H, because $N(S_\mu) = V(G) - S$ and S_μ is stable. In other words, $S_\mu \in \Psi(G)$. Since H is bipartite and M is its unique perfect matching, Proposition 3.1 implies that there exists a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{\mu-1}\} \subset \{x_1, x_2, ..., x_{\mu-1}, x_\mu\} = S_\mu,$$

such that all $S_k = \{x_1, x_2, ..., x_k\}, 1 \leq k \leq \mu$ are local maximum stable sets in H. The equality $N_H[S_k] = N_G[S_k]$ explains why $S_k \in \Psi(G)$ for all $k \in \{1, ..., \mu(G)\}$. Let now $x \in S - S_\mu$. Then $N(x) \subseteq V(G) - S$, and therefore, $N(S_\mu \cup \{x\}) = V(G) - S$. Since S_μ is a maximum stable set in H and $S_\mu \cup \{x\}$ is stable in $H \cup \{x\} = G[N[S_\mu \cup \{x\}]]$, we get that $S_\mu \cup \{x\}$ is a maximum stable set in $H \cup \{x\}$, i.e., $S_{\mu+1} = S_\mu \cup \{x\} \in \Psi(G)$. If there still exists some $y \in S - S_{\mu+1}$, in the same manner as above we infer that $S_{\mu+2} = S_{\mu+1} \cup \{y\} \in \Psi(G)$.

In such a way we build the following accessibility chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_\mu\} \subset S_{\mu+1} \subset S_{\mu+1} \subset ... \subset S_\alpha = S.$$

Clearly, $(iii) \Rightarrow (i)$, and this completes the proof. ■

As an example of the process of building a uniquely restricted maximum matching with the help of an accessibility chain, let us consider the bipartite graph G presented in Figure 6. The accessibility chain

$$\{h\} \subset \{h, d\} \subset \{h, d, f\} \subset \{h, d, f, c\} \subset \{h, d, f, c, a\} \in \Psi(G)$$

gives rise to the uniquely restricted maximum matching $M = \{hg, de, cb\}$. Notice that $\Psi(G)$ is not a greedoid, because $\{d, f\} \in \Psi(G)$, while $\{d\}, \{f\} \notin \Psi(G)$.

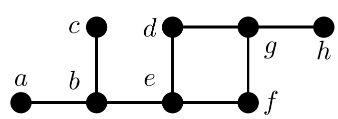

Figure 6: The chain of uniquely restricted matchings is :$\{hg\}, \{hg, de\}, \{hg, de, cb\}$.

The following theorem will show us another reason, why the family $\Psi(G)$ of the graph G from Figure 6 is not a greedoid, namely $\{bc, de, fg\}$ is a maximum matching, but not uniquely restricted.

Theorem 3.3 *If G is a bipartite graph, then $\Psi(G)$ is a greedoid if and only if all its maximum matchings are uniquely restricted.*

Proof. Assume that $\Psi(G)$ is a greedoid. Let M be a maximum matching in G. According to Lemma 2.2, we have that $M \subseteq (S, V(G) - S)$ and $|M| = |V(G) - S|$ for any $S \in \Omega(G)$. Let S_μ contain the vertices of some $S \in \Omega(G)$ matched by M with the vertices of $V(G) - S$. Since M is a perfect matching in $G[N[S_\mu]]$ and $|S_\mu| = |M|$, it follows that S_μ is a maximum stable set in $G[N[S_\mu]]$, i.e., $S_\mu \in \Psi(G)$. Hence, there exists an accessibility chain of the following structure:

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_\mu\} = S_\mu \subset S_\mu \cup \{x_{\mu+1}\} \subset ... \subset S.$$

While the existence of the first part of this chain, i.e., $\{x_1\},\{x_1,x_2\},...,\{x_1,x_2,...,x_\mu\}$, is based on the accessibility property of the family $\Psi(G)$, the existence of the second part of the same chain, namely $S_\mu, S_\mu \cup \{x_{\mu+1}\},...,S$, stems from the exchange property of $\Psi(G)$. Now, according to Proposition 3.1, we may conclude that the perfect matching M is unique in $G[N[S_\mu]]$. Hence, M is a uniquely restricted maximum matching in G.

Conversely, suppose that all maximum matchings of G are uniquely restricted. Let $\widehat{S} \in \Psi(G), H = G[N[\widehat{S}]]$, and $\widehat{M}$ be a maximum matching in H. The graph H is bipartite as a subgraph of a bipartite graph. By Lemma 2.3, there exists a maximum matching in G, say M, such that $\widehat{M} \subseteq M$. Since M is uniquely restricted in G, it follows that $\widehat{M}$ is uniquely restricted in H. According to Theorem 3.2, there exists an accessibility chain of $\widehat{S}$ in H

$$S_1 \subset S_2 \subset ... \subset S_{q-1} \subset S_q = \widehat{S}.$$

Since $N_H[S_k] = N_G[S_k]$, we infer that $S_k \in \Psi(G)$, for any $k \in \{1,...,q\}$.

To complete the proof, we have to show that, in addition to the accessibility property, $\Psi(G)$ satisfies also the exchange property.

Let $X, Y \in \Psi(G)$ and $|Y| = |X|+1 = m+1$. Hence, there is an accessibility chain

$$\{y_1\} \subset \{y_1,y_2\} \subset ... \subset \{y_1,...,y_m\} \subset \{y_1,...,y_m,y_{m+1}\} = Y.$$

Since Y is stable, $X \in \Psi(G)$, and $|X| < |Y|$, it follows that there exists some $y \in Y - X$, such that $y \notin N[X]$. Let M_X be a maximum matching in $H = G[N[X]]$. Since H is bipartite, X is a maximum stable set in H, and M_X is a maximum matching in H, it follows that

$$|X| + |M_X| = |N[X]| = |X| + |N(X)|\,,\ \textit{i.e.}, |M_X| = |N(X)|\,.$$

Let $y_{k+1} \in Y$ be the first vertex in Y satisfying the conditions: $y_1,...,y_k \in N[X]$ and $y_{k+1} \notin N[X]$. Since $\{y_1,...,y_k\}$ is stable in $N[X]$, there is $\{x_1,...,x_k\} \subseteq X$ such that for any $i \in \{1,...,k\}$ either $x_i = y_i$ or $x_iy_i \in M_X$.

Now we show that $X \cup \{y_{k+1}\} \in \Psi(G)$.

Case 1. $N[X \cup \{y_{k+1}\}] = N[X] \cup \{y_{k+1}\}$. Clearly, $X \cup \{y_{k+1}\}$ is stable in $G(N[X \cup \{y_{k+1}\}])$ and $|X \cup \{y_{k+1}\}| = |X| + 1$ ensures that $X \cup \{y_{k+1}\} \in \Psi(G)$, because $X \in \Psi(G)$ too.

Case 2. $N[X \cup \{y_{k+1}\}] \neq N[X] \cup \{y_{k+1}\}$. Suppose there are $a, b \in N(y_{k+1}) - N[X]$. Hence, it follows that $\{a,b,x_1,...,x_k\}$ is a stable set included in $N[\{y_1,...,y_{k+1}\}]$ and larger than $\{y_1,...,y_{k+1}\}$, in contradiction with the fact that $\{y_1,...,y_{k+1}\} \in \Psi(G)$. Therefore, there exists a unique $a \in N(y_{k+1}) - N[X]$. Consequently,

$$N[X \cup \{y_{k+1}\}] = N[X] \cup N[y_{k+1}] = N[X] \cup \{a, y_{k+1}\},$$

and since $ay_{k+1} \in E(G)$, we obtain that $X \cup \{y_{k+1}\}$ is a maximum stable set in $G[N[X \cup \{y_{k+1}\}]]$, i.e., $X \cup \{y_{k+1}\} \in \Psi(G)$. ■

As an immediate consequence of Theorem 3.3, we obtain the following:

Corollary 3.4 *For any bipartite graph G having a perfect matching, $\Psi(G)$ is a greedoid if and only if G has a unique perfect matching.*

Corollary 3.4 and, consequently, Theorem 3.3 are not valid for non-bipartite graphs. For example, the graph C_5+e in Figure 7 is a non-bipartite graph having only uniquely restricted maximum matchings, (in fact, it has a unique perfect matching), but $\Psi(C_5+e)$ is not a greedoid, because $\{u,v\} \in \Psi(C_5+e)$, while $\{u\},\{v\} \notin \Psi(C_5+e)$.

Figure 7: Non-bipartite graphs with unique perfect matchings.

However, there are non-bipartite graphs with a unique perfect matching, whose $\Psi(G)$ is a greedoid. For instance, while the graph C_5+3e in Figure 7 is a non-bipartite graph with a unique perfect matching, the family $\Psi(C_5+3e)$ is a greedoid.

Let us also notice that there exist both bipartite and non-bipartite graphs without a perfect matching whose family of local maximum stable sets is a greedoid. For instance, neither G_1 nor G_2 in Figure 8 has a perfect matching, G_1 is bipartite, and $\Psi(G_1), \Psi(G_2)$ are greedoids.

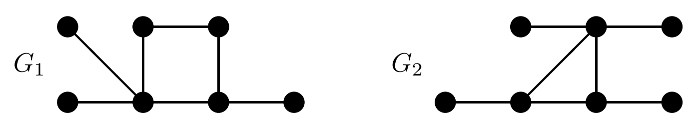

Figure 8: $\Psi(G_1)$ and $\Psi(G_2)$ form greedoids, but only G_1 is a bipartite graph.

Since any forest, by definition, has no cycles, the following Lemma 3.5 ensures that all matchings of a forest are uniquely restricted.

Lemma 3.5 *[8] If a bipartite graph has two perfect matchings M_1 and M_2, then any of its vertices, from which are issuing edges contained in M_1 and M_2, respectively, belongs to some cycle that is alternating with respect to at least one of M_1, M_2.*

It is also interesting to note that Golumbic, Hirst and Lewenstein have proved the following generalization of Lemma 3.5.

Theorem 3.6 *[4] A matching M in a graph G is uniquely restricted if and only if there is no even-length cycle with edges alternating between matched and non-matched edges.*

Now restricting Theorem 3.3 to forests we immediately obtain that the family of local maximum stable sets of a forest forms a greedoid on its vertex set, which gives a new proof of the main finding from [10], namely Theorem 1.2.

4 Conclusions

We have shown that to have all maximum matchings uniquely restricted is necessary and sufficient for a bipartite graph G to enjoy the property that $\Psi(G)$ is a greedoid. We have also described all the bipartite graphs having a unique perfect matching, or in other words, all bipartite graphs having a perfect matching and whose $\Psi(G)$ is a greedoid. It seems to be interesting to describe a recursive structure of all bipartite graphs whose $\Psi(G)$ is a greedoid.

A linear time algorithm to decide whether a matching in a bipartite graph is uniquely restricted is presented in [4]. It is also shown there that the problem of finding a maximum uniquely restricted matching is **NP**-complete for bipartite graphs. These results motivate us to propose another open problem, namely: how to recognize bipartite graphs whose $\Psi(G)$ is a greedoid?

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